

Full diversity sets of unitary matrices from orthogonal sets of idempotents*

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Abstract

Orthogonal sets of idempotents are used to design sets of unitary matrices, known as constellations, such that the modulus of the determinant of the difference of any two distinct elements is greater than 0. It is shown that unitary matrices in general are derived from orthogonal sets of idempotents reducing the design problem to a construction problem of unitary matrices from such sets. The quality of the constellations constructed in this way and the actual differences between the unitary matrices can be determined algebraically from the idempotents used. This has applications to the design of unitary space time constellations.

1 Introduction

The design problem for unitary space time constellations is set out nicely in [1] and [4]: “Let M be the number of transmitter antennas and R the desired transmission rate. Construct a set \mathcal{V} of $L = 2^{RM}$ unitary $M \times M$ matrices such that for any two distinct elements A, B in \mathcal{V} , the quantity $|\det(A - B)|$ is as large as possible. Any set \mathcal{V} such that $|\det(A - B)| > 0$ for all distinct A, B is said to have *full diversity*.”

The number of transmitter antennas is the size M of the matrices and this is also known as the order of the constellation or matrices. ‘Order’ in this instance refers to the size of the matrices.

The set \mathcal{V} is known as a *constellation*. In [1] also it is explained that the *quality* of the constellation is measured by

$$\zeta_{\mathcal{V}} = \frac{1}{2} \min_{V_l, V_m \in \mathcal{V}, V_l \neq V_m} |\det(V_l - V_m)|^{\frac{1}{M}}$$

Here we present general methods for constructing such constellations from orthogonal sets of idempotents. It is shown that unitary matrices are obtained from complete orthogonal sets of idempotents in a precise manner. This enables constructions of constellations using such representations and the nature of the constructions allows the quality to be determined algebraically; all differences may often be explicitly calculated.

New constellation are derived from the general concept, explicit constructions are given and many more may be derived. Indeed infinite series of fully diverse real and infinite series of fully diverse complex constellations may be constructed using the methods; from these finite sets may be chosen and the quality worked out algebraically as required.

Extension methods for constructing constellations are derived. Algebraic results on differences of unitary matrices are formulated which may then be used to calculate the quality of such constructed constellations.

A method is derived in Section 5 which allows the construction of constellations of order $2n \times 2n$ from constellations of order $n \times n$ where the higher order constellations have similar quality and similar rate to the lower order constellations. In this way many more constellations of higher order may be constructed from those already constructed.

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Examples are constructed which show some of the range of the methods but the methods are fairly general and many more may be constructed.

Division algebras have also been used in this area and the excellent survey article [5] and the references therein give the details. See also [2], [3].

1.1 Further notation

For unitary matrices A, B of the same $M \times M$ size define the *distance* or *difference* between A and B to be $\frac{1}{2}|\det(A - B)|^{\frac{1}{M}}$. Thus for a constellation \mathcal{V} of unitary matrices consisting of $M \times M$ matrices its quality is the minimum of the distances between any two different matrices in \mathcal{V} .

A set of orthogonal idempotents in a ring R is a set $\{e_1, e_2, \dots, e_k\}$ satisfying:

(i) $e_i \neq 0$ and $e_i^2 = e_i$, $1 \leq i \leq k$.

(ii) If $i \neq j$ then $e_i e_j = 0$.

If further $1 = e_1 + e_2 + \dots + e_k$ then the set is said to be a *complete set of orthogonal idempotents*.

Here we use 1 for the identity of R . In general 1 will denote the identity of the system under consideration.

The idempotent e_i is said to be *primitive* if it cannot be written as $e_i = e'_i + e''_i$ where e'_i, e''_i are idempotents such that $e'_i, e''_i \neq 0$ and $e'_i e''_i = 0$. A set of orthogonal idempotents is said to be *primitive* if each idempotent in the set is primitive.

A mapping $*$: $R \rightarrow R$ in which $r \mapsto r^*$, ($r \in R$) is said to be an *involution* on R if and only if (i) $r^{**} = r$, $\forall r \in R$, (ii) $(a + b)^* = a^* + b^*$, $\forall a, b \in R$, and (iii) $(ab)^* = b^* a^*$, $\forall a, b \in R$.

We are particularly interested in the case where $*$ denotes complex conjugate transpose in the case of matrices over \mathbb{C} and denotes transpose for matrices over other fields and in particular over \mathbb{R} , the reals.

If R has an involution $*$ then an element $v \in R$ is said to be *symmetric* (with respect to $*$) if $v^* = v$ and a set of elements is said to be symmetric if each element in the set is symmetric.

The matrix $U \in R_{n \times n}$ is said to be a unitary matrix (with respect to $*$) if $UU^* = 1$.

A constellation is said to be *fully diverse* when it has full diversity.

Further general algebra background may be found [6] although little background in coding theory itself is required.

1.2 Layout

In Section 2 the connection between orthogonal sets of idempotents and unitary matrices is established and in the (sub)Section 2.2 properties of, and construction methods for, orthogonal sets of idempotent matrices are analysed.

In Section 3 methods are derived for constructing unitary matrices from a complete set of idempotents. The results here correspond to those obtained in the cyclic case as in [1] and [4]. Examples are given and the rates and quality are worked out.

Methods are derived for constructing and analysing unitary matrices using different sets of orthogonal idempotents in Section 4. In (sub)Sections 4.1, 4.2, 4.3 the methods are applied to constructing infinite fully diverse sets of real unitary and their distances and quality of these sets are then known. Examples are given here and many more may be deduced. In Section 4.6 the methods are applied to constructing sets of constellations with complex entries; indeed infinite such sets are constructed from which finite subsets may be deduced as required.

In Section 5, a method, using what is called a tangle of matrices, is devised to construct sets of $2n \times 2n$ fully diverse constellations from a set of $n \times n$ fully diverse constellations. The quality of the $2n \times 2n$ constellations may be given in terms of the quality of the $n \times n$ constellations from which they are derived.

1.3 Dependence

Some of the sections may be read independently except where a reference is made to an example constructed in a previous section. In this sense Sections 3, 4, 5 may be read independently. The (sub)Sections 2.2 and 2.3, on methods for constructing orthogonal sets of idempotent matrices and properties therefrom, may be consulted as required.

1.4 Determinants of block matrices

Interested will be in $P = \det\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C, D are block matrices of the same size. It is not necessary that all of A, B, C, D commute in order to have a formula (such as below) for P in terms of A, B, C, D .

$$\text{Let } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then

1. $\det M = \det(AD - BC)$ whenever at least one of A, B, C, D is equal to the zero matrix.
2. $\det M = \det(AD - BC)$ when $DC = CD$.
3. $\det M = \det(AD - CB)$ when $AC = CA$.
4. $\det M = \det(DA - BC)$ when $BD = DB$.
5. $\det M = \det(DA - CB)$ when $AB = BA$.

Such results may be found on-line or in for example [7]. These will be applied without further reference.

2 Unitary matrices and orthogonal sets of idempotents

Unitary matrices over \mathbb{C} are built from complete symmetric orthogonal sets of matrices as follows:

Proposition 2.1 *U is a unitary $n \times n$ matrix over \mathbb{C} if and only if $U = \alpha_1 v_1^* v_1 + \alpha_2 v_2^* v_2 + \dots + \alpha_n v_n^* v_n$ where $\{v_1^* v_1, v_2^* v_2, \dots, v_n^* v_n\}$ is a complete symmetric orthogonal set of idempotents in $\mathbb{C}_{n \times n}$ and $\alpha_i \in \mathbb{C}$ with $|\alpha_i| = 1, \forall i$. Further the α_i are the eigenvalues of U .*

This result appears in [10, 9] but as it leads to fundamental constructions, a proof is given here for completeness.

Proof: Let $U = \alpha_1 v_1^* v_1 + \alpha_2 v_2^* v_2 + \dots + \alpha_n v_n^* v_n$ where $\{v_1^* v_1, v_2^* v_2, \dots, v_n^* v_n\}$ is a orthogonal complete set of idempotents with $|\alpha_i| = 1$. It is easy to check that $UU^* = 1$. Then $Uv_i^* = \alpha_i v_i^*$ and so the α_i are the eigenvalues of U .

Suppose then U is a unitary matrix. It is known, as in particular U is a normal matrix, that there exists a unitary matrix P such that $U = P^* D P$ where D is diagonal and the entries of D must have modulus 1. Thus $P = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ where $\{v_1, v_2, \dots, v_n\}$ is an orthonormal basis (of row vectors) for \mathbb{C}_n and $D = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$ with $|\alpha_i| = 1$ and the α_i are the eigenvalues of U . Then

$$\begin{aligned} U &= P^* D P \\ &= (v_1^*, v_2^*, \dots, v_n^*) \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \\ &= (\alpha_1 v_1^*, \alpha_2 v_2^*, \dots, \alpha_n v_n^*) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \\ &= \alpha_1 v_1^* v_1 + \alpha_2 v_2^* v_2 + \dots + \alpha_n v_n^* v_n. \end{aligned}$$

□

Thus unitary matrices are generated by complete symmetric orthogonal sets of idempotents formed from the diagonalising unitary matrix. Notice that the α_i are the eigenvalues of U .

2.1 Example

For example consider the real orthogonal/unitary matrix $U = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$. This has eigenvalues $e^{i\theta}, e^{-i\theta}$ and $P = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -i \\ i & 1 \end{pmatrix}$ is a diagonalising unitary matrix. Take the rows $v_1 = \frac{1}{\sqrt{2}}(-1, -i)$, $v_2 = \frac{1}{\sqrt{2}}(i, 1)$ of P and consider the complete orthogonal symmetric set of idempotents $\{P_1 = v_1^* v_1 = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, P_2 = v_2^* v_2 = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\}$.

Then applying Proposition 2.1 gives $U = e^{i\theta} P_1 + e^{-i\theta} P_2 = \frac{1}{2} e^{i\theta} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + \frac{1}{2} e^{-i\theta} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$, which may be checked independently to be equal to U .

2.2 Complete orthogonal sets of idempotents

Unitary matrices are designed from complete symmetric sets of orthogonal idempotents as in Proposition 2.1.

Also in [10, 9] construction methods for complete symmetric orthogonal systems of idempotents are given. The methods are based essentially on (a) orthogonal projections; (b) group rings. The reader may consult the results in these papers as required later.

Methods similar have been used to construct series of *paraunitary matrices* which play an important role in signal processing, [10].

2.3 Rank and Determinants

The results in this subsection are used later for constructing constellations and for calculations the differences and quality. They appear essentially in [10, 9] but in a slightly different form.

Lemma 2.1 Suppose $\{E_1, E_2, \dots, E_s\}$ is a set of orthogonal idempotent matrices. Then $\text{rank}(E_1 + E_2 + \dots + E_s) = \text{tr}(E_1 + E_2 + \dots + E_s) = \text{tr} E_1 + \text{tr} E_2 + \dots + \text{tr} E_s = \text{rank } E_1 + \text{rank } E_2 + \dots + \text{rank } E_s$.

Proof: It is known that $\text{rank } A = \text{tr} A$ for an idempotent matrix, see for example [11], and so $\text{rank } E_i = \text{tr} E_i$ for each i . If $\{E, F, G\}$ is a set of orthogonal idempotent matrices so is $\{E + F, G\}$. From this it follows that $\text{rank}(E_1 + E_2 + \dots + E_s) = \text{tr}(E_1 + E_2 + \dots + E_s) = \text{tr} E_1 + \text{tr} E_2 + \dots + \text{tr} E_s = \text{rank } E_1 + \text{rank } E_2 + \dots + \text{rank } E_s$. □

Corollary 2.1 $\text{rank}(E_{i_1} + E_{i_2} + \dots + E_{i_k}) = \text{rank } E_{i_1} + \text{rank } E_{i_2} + \dots + \text{rank } E_{i_k}$ for $i_j \in \{1, 2, \dots, s\}$, $i_j \neq i_l$.

Let $\{e_1, e_2, \dots, e_k\}$ be a complete orthogonal set of idempotents in a vector space over F .

Theorem 2.1 Let $w = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_k e_k$ with $\alpha_i \in F$. Then w is invertible if and only if each $\alpha_i \neq 0$ and in this case $w^{-1} = \frac{1}{\alpha_1} e_1 + \frac{1}{\alpha_2} e_2 + \dots + \frac{1}{\alpha_k} e_k$.

Proof: Suppose each $\alpha_i \neq 0$. Then $w(\frac{1}{\alpha_0} e_0 + \frac{1}{\alpha_1} e_1 + \dots + \frac{1}{\alpha_k} e_k) = e_0^2 + e_1^2 + \dots + e_k^2 = e_0 + e_1 + \dots + e_k = 1$.

Suppose w is invertible and that some $\alpha_i = 0$. Then $w e_i = 0$ and so w is a (non-zero) zero-divisor and is not invertible. □

We now specialise the e_i to be $n \times n$ matrices and in this case use capital letters and let $e_i = E_i$.

Let $A = a_1 E_1 + a_2 E_2 + \dots + a_k E_k$. Then A is invertible if and only if each $a_i \neq 0$ and in this case $A^{-1} = \frac{1}{a_1} E_1 + \frac{1}{a_2} E_2 + \dots + \frac{1}{a_k} E_k$.

The following result is very useful for determining the quality of constellations constructed by the methods of idempotents.

Theorem 2.2 Suppose $\{E_1, E_2, \dots, E_k\}$ is a complete symmetric orthogonal set of idempotents in $F_{n \times n}$. Let $A = a_1 E_1 + a_2 E_2 + \dots + a_k E_k$. Then the determinant of A is $|A| = a_1^{\text{rank } E_1} a_2^{\text{rank } E_2} \dots a_k^{\text{rank } E_k}$.

Proof: Now $AE_i = a_i E_i^2 = a_i E_i$. Thus each column of E_i is an eigenvector of A corresponding to the eigenvalue a_i . Thus there are at exist $\text{rank } E_i$ linearly independent eigenvectors corresponding to the eigenvalue a_i . Since $\text{rank } E_1 + \text{rank } E_2 + \dots + \text{rank } E_k = n$ there are exactly $\text{rank } E_i$ linearly independent eigenvectors corresponding to the eigenvalue a_i . Let $r_i = \text{rank } E_i$. Let these r_i linearly independent eigenvectors corresponding to a_i be denoted by $v_{i,1}, v_{i,2}, \dots, v_{i,r_i}$. Do this for each i .

Any column of E_i is perpendicular to any column of E_j for $i \neq j$ as $E_i E_j^* = 0$.

Suppose now $\sum_{j=1}^{r_1} \alpha_{1,j} v_{1,r_j} + \sum_{j=1}^{r_2} \alpha_{2,j} v_{2,r_j} + \dots + \sum_{j=1}^{r_k} \alpha_{k,j} v_{k,j} = 0$.

Multiply through by E_s for $1 \leq s \leq k$. This gives $\sum_{j=1}^{r_k} \alpha_{k,j} v_{k,j} = 0$ from which it follows that $\alpha_{k,j} = 0$ for $j = 1, 2, \dots, r_k$.

Thus the set of vectors $S = \{v_{1,1}, v_{1,2}, \dots, v_{1,r_1}, v_{2,1}, v_{2,2}, \dots, v_{2,r_2}, \dots, v_{k,1}, v_{k,2}, \dots, v_{k,r_k}\}$ is linearly independent and form a basis for F^n – remember that $\text{rank}(E_1 + E_2 + \dots + E_k) = n$. Hence A can be diagonalised by the matrix of these vectors and thus there is a non-singular matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix consisting of the a_i repeated r_i times for each $i = 1, 2, \dots, k$.

Hence $|A| = |D| = a_1^{r_1} a_2^{r_2} \dots a_k^{r_k}$. \square

Theorem 2.2 may be used to compute the full distribution of the differences in a constellation in certain cases.

The following Proposition may be found in [10], Proposition 4.2.

Proposition 2.2 *Let F be a field in which every element has a square root. Suppose also an involution $*$ is defined on the set of matrices over F . Then P is a symmetric (with respect to $*$) idempotent of rank 1 in $F_{n \times n}$ if and only if $P = vv^*$ where v is a column vector such that $v^*v = 1$.*

(Note that ‘symmetric with respect to $*$ ’ in the case of matrices over \mathbb{C} is usually termed ‘Hermitian’.)

It is necessary that square roots exist in the field and an example is given in [10] to demonstrate this.

Proposition 2.2 shows that vv^* , for v a unit column vector, is a symmetric idempotent and sets of unitary matrices are constructed from these types of idempotents in later sections.

3 Constellations from complete orthogonal set of idempotents

Recall that a set of unitary matrices is said to have *full diversity* or to be fully diverse if and only if the modulus of the determinant of the difference of any two matrices in the set is non-zero.

Theorem 3.1 *Let $\{E_1, E_2, \dots, E_k\}$ be a complete symmetric orthogonal set of idempotents. Define for $s = 1, 2, \dots, t$, $V_s = \sum_{j=1}^k \alpha_{s,j} E_j$ where the $\alpha_{p,q}$ are complex numbers of modulus 1. Then $\mathcal{V} = \{V_1, V_2, \dots, V_t\}$ is a constellation of unitary matrices. Further \mathcal{V} has full diversity if and only if for each $t = 1, 2, \dots, k$, $\alpha_{r,t} \neq \alpha_{s,t}$ when $r \neq s$.*

Proof: It is easy to check that each V_j is a unitary matrix. Consider all $W(p, l) = V_p - V_l$ for $p \neq l$. Then $W(p, l) = \sum_{i=1}^k (\alpha_{p,i} - \alpha_{l,i}) E_i$ and by Theorem 2.2, $\det(W(p, l)) = (\alpha_{p,1} - \alpha_{l,1})^{\text{rank } E_1} (\alpha_{p,2} - \alpha_{l,2})^{\text{rank } E_2} \dots (\alpha_{p,k} - \alpha_{l,k})^{\text{rank } E_k}$. Hence $|\det(W(p, l))| \neq 0$ for all p, l with $p \neq l$ if and only if $\alpha_{p,i} \neq \alpha_{l,i}$ for $1 \leq i \leq k$ and for all p, l with $p \neq l$. \square

Theorem 3.1 enables the construction of classes of constellations, and Theorem 2.2 enables the calculation of the quality of each one and indeed the calculation of all the differences in the constellation.

3.1 Examples

Theorem 3.1 is now used to construct constellations and calculate their qualities. To keep full diversity it is only necessary to adhere to the conditions of Theorem 3.1.

Consider the 2×2 case. Let $\{E_0, E_1\}$ be a complete orthogonal set of idempotents in $\mathbb{C}_{2 \times 2}$.

For fixed θ , use (k, j) to mean $e^{ki\theta} E_0 + e^{ji\theta} E_1$. Use $|(p, q)|$ to mean $|1 - e^{ip\theta}| |1 - e^{iq\theta}|$. Note that $|1 - e^{i\alpha}| = |1 - e^{-i\alpha}|$.

1. $n = 5, \theta = 2\pi/5$: $\mathcal{V} = \{(0, 2), (1, 4), (2, 1), (3, 3), (4, 0)\}$. Then quality is $(\sin(\pi/5) \sin(2\pi/5))^{1/2} = 0.74767\dots$. The rate here is $\log_2(5)/2 = 1.1609\dots$
2. $n = 8$: $\mathcal{V} = \{(0, 0), (1, 3), (2, 6), (3, 1), (4, 4), (5, 7), (6, 2), (7, 5)\}$. This is the form $\{(j, 3j) \mid j = 0, 1, \dots, 7\}$ where $3j$ is interpreted as $3j \bmod 8$. Here the modulus of difference between any pairs of these is at worst $|(2, 2)|$ or $|(1, 3)|$. It is easy to see that $|(2, 2)| > |(1, 3)|$. Then quality is $(\sin(\pi/8) \sin(3\pi/8))^{1/2} = 0.5946\dots$. The rate is $\log_2(8)/2 = 1.5$.
It is possible here also to determine the distribution of the differences. There are in total 28 differences between the 8 elements of the constellations. These are distributed as follows. There are 16 of modulus $|\sin(\pi/8) \sin(3\pi/8)|^{1/2} = 0.5946\dots$, 8 of modulus $\sin(2\pi/8) \sin(2\pi/8)^{1/2} = 0.707$ and 4 of modulus $(\sin(4\pi/8) \sin(4\pi/8))^{1/2} = 1$. A weighted average is $0.684657\dots$, which may possibly be a more correct measure of a ‘quality’ subject to the main quality.
3. $n = 32$. Let the constellation be $\{(j, 7j \bmod 32) \mid j = 0, 1, \dots, 31\}$. On noting that $|(5, 3)| > |(1, 7)|$ it is seen that the quality of this is $\frac{1}{2}(|1 - e^{i\theta}| |1 - e^{7i\theta}|)^{1/2} = (\sin(\pi/32) \sin(7\pi/32))^{1/2}$ and this is $0.24936\dots$. Here also the distribution of the differences may be determined.
4. $n = 64$. Use the constellation $\{(i, 19i) \mid i = 0, 1, \dots, 63\}$ with $19i$ interpreted as $19i \bmod 64$. Then get quality $(\sin(\pi/64) \sin(19\pi/64))^{1/2} = 0.1985\dots$. Note here that $|(10, 2)| > |(19, 1)|$. The rate here is $\log_2(64)/2 = 3$.
5. $n = 128$. Use the constellation $\{(i, 47i) \mid i = 0, 1, \dots, 127\}$. Then get quality $(\sin(\pi/128) \sin(47\pi/128))^{1/2} = 0.14978\dots$. There are other pairs $(i, 47i)$ where $47i \bmod 128 < 47$ but here the i is big enough so that $|(i, 47i)| > |(1, 47)|$. The rate here is $\log_2(128)/2 = 3.5$.

3.2 Higher order

Let $\{E_0, E_1, E_2\}$ be a complete orthogonal set of idempotents in $C_{3 \times 3}$. Use (k, l, m) to mean $e^{ki\theta} E_0 + e^{li\theta} E_1 + e^{mi\theta} E_2$ when θ has been given a value.

Suppose now $n = 8$ and $e^{i\theta} = e^{2i\pi/8}$ is a primitive 8^{th} root of 1. Consider the constellation:

$$\{(0, 0, 7), (1, 3, 2), (2, 6, 5), (3, 1, 3), (4, 4, 0), (5, 7, 1), (6, 2, 6), (7, 5, 4)\}.$$

This has quality $(\sin(\pi/8) \sin(3\pi/8) \sin(\pi/8))^{1/3} = 0.51337\dots$. There are just two of the 28 differences with this least modulus. All the other differences have modulus at least $(\sin(2\pi/8) \sin(2\pi/8) \sin(\pi/8))^{1/3} = 0.5762\dots$.

For even order (i.e. for $C_{2k \times 2k}$) by simply repeating the 2 distribution it is possible to obtain the same quality as the 2×2 case but with smaller rate. Improvements on this can also be obtained by matching ‘bad’ pairs with ‘good’ pairs.

For example consider a complete orthogonal set of idempotents $\{E_1, E_2, E_3, E_4\}$ in $C_{4 \times 4}$. Then repeat the pattern for the 2×2 case. Let $\theta = 2\pi/8$ and define $(j, k, l, m) = e^{ji\theta} E_1 + e^{ki\theta} E_2 + e^{li\theta} E_3 + e^{mi\theta} E_4$. Define the constellation $\{(j, 3j, j, 3j) \mid j = 0, 1, 2, \dots, 7\}$ where $3j$ is interpreted as $3j \bmod 8$. Then the quality of this constellation is also $(\sin^2(\pi/8) \sin^2(3\pi/8))^{1/4} = 0.5946\dots$. With modification to the construction the quality can be improved to $(\sin^2(\pi/8) \sin(3\pi/8) \sin(4\pi/8))^{1/4} = 0.60649\dots$.

3.3 General construction

In general consider a complete orthogonal set of idempotents $\{E_1, E_2, \dots, E_s\}$ in $C_{n \times n}$. Let $e^{2\pi/k} = e^{i\theta}$ be a primitive k^{th} root of 1. Define $(j_1, j_2, \dots, j_k) = \sum_{t=1}^s e^{j_t i \theta} E_t$. Then construct the constellation $\{(j_{i1}, j_{i2}, \dots, j_{is}) \mid i = 1, 2, \dots, r, 1 \leq r \leq k\}$ where, for each i , $j_{ti} \neq j_{qi}$ for $t \neq q$. Maximise the quality by choosing the j_{kl} to maximise the minimum modulus of the differences.

This could be further developed but is left for consideration elsewhere.

4 Constellations combining different orthogonal sets of idempotents

Now consider constructing constellations by combining different sets of orthogonal idempotents. Good constellations and indeed good constellations with real unitary matrices and good quality can still be obtained.

4.1 Symmetric 2×2 idempotents

Let $\{P, P_1\}$ be a complete symmetric orthogonal set of idempotents in $\mathbb{C}_{2 \times 2}$. Then $P_1 = 1 - P$ and since P is a symmetric idempotent it follows that $P = vv^*$ for a unit row-vector v by Proposition 2.2. Let $v = \begin{pmatrix} a \\ b \end{pmatrix}$ and thus $P = vv^* = \begin{pmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{pmatrix}$ where $|a|^2 + |b|^2 = 1$.

Consider also the idempotent $Q = \begin{pmatrix} |c|^2 & cd^* \\ dc^* & |d|^2 \end{pmatrix}$ where $|c|^2 + |d|^2 = 1$.

Proposition 4.1 *Let $P = \begin{pmatrix} a \\ b \end{pmatrix} (a^*, b^*)$ and $Q = \begin{pmatrix} c \\ d \end{pmatrix} (c^*, d^*)$. Then $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a^* & b^* \\ -c^* & -d^* \end{pmatrix} = P - Q$.*

Proof: This may be shown directly by matrix multiplication. □

Corollary 4.1 $|\det(P - Q)| = |\det\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a^* & b^* \\ -c^* & -d^* \end{pmatrix}\right)| = |\det\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right)|^2$

Proof: This follows since $|\det X| = |\det(-X)| = |\det X^*|$ for a matrix X . □

Corollary 4.2 $|\det(2(P - Q))| = 4|ad - bc|^2$.

Corollary 4.3 $|\det(P - Q)| = 0$ if and only if $ad = bc$.

Corollary 4.4 When a, b, c, d are real, $|\det(2(P - Q))| = 4(ad - bc)^2$.

The following more general result is needed later.

Lemma 4.1 $\begin{pmatrix} aa^* - \alpha cc^* & ab^* - \alpha cd^* \\ ba^* - \alpha dc^* & bb^* - \alpha dd^* \end{pmatrix} = \begin{pmatrix} a & \sqrt{\alpha}c \\ b & \sqrt{\alpha}d \end{pmatrix} \begin{pmatrix} a^* & b^* \\ -\sqrt{\alpha}c^* & -\sqrt{\alpha}d^* \end{pmatrix}$.

Proof: This follows by direct matrix multiplication. □

Corollary 4.5 Suppose $E = \begin{pmatrix} aa^* & ab^* \\ ba^* & bb^* \end{pmatrix}$, $F = \begin{pmatrix} cc^* & cd^* \\ dc^* & dd^* \end{pmatrix}$. Then $|\det(E - F)| = |\det\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right)|^2$ and $|\det(E - \alpha F)| = |\alpha \det(E - F)|$.

Proof: That $|\det(E - F)| = |\det\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}\right)|^2$ follows from Corollary 4.1.

Now $|\det(E - \alpha F)| = |\det\left(\begin{pmatrix} a & \sqrt{\alpha}c \\ b & \sqrt{\alpha}d \end{pmatrix} \begin{pmatrix} a^* & b^* \\ -\sqrt{\alpha}c^* & -\sqrt{\alpha}d^* \end{pmatrix}\right)| = |\sqrt{\alpha}\sqrt{\alpha} \det\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a^* & b^* \\ -c^* & -d^* \end{pmatrix}\right)| = |\alpha \det(E - F)|$. □

Now $U = P - P_1 = 2P - I$ and $V = Q - Q_1 = 2Q - I$ are unitary matrices and $U - V = 2(P - Q)$. Note that U, V do not in general commute so cannot be simultaneously diagonalised. Thus these constructions are not the same as the cyclic constructions.

More generally we have the following result.

Lemma 4.2 Let E be a symmetric idempotent in $\mathbb{C}_{n \times n}$. Then $2E - I_n$ is a unitary matrix.

Proof: Write I for I_n . Then $(2E-I)(2E-I)^* = (2E-I)(2E-I) = 4E^2 - 2E - 2E + I = 4E - 4E + I = I$. \square

Now form constellations of the form $\{2E_i - I | i \in J\}$ for some index set J where E_i are symmetric idempotents. These matrices do not commute in general so such constellations are certainly different to those in Section 3 and are for example cyclic.

Suppose the matrices are of order 2×2 . The differences of the elements in the constellation are $\frac{1}{2} |(\det(2E_i - I) - \det(2E_j - I))|^{1/2} = \frac{1}{2} |\det 2(E_i - E_j)|^{1/2} = \frac{1}{2} |4 \det(E_i - E_j)|^{1/2} = |\det(E_i - E_j)|^{1/2}$. These can be calculated by results such as Corollary 4.2 for certain idempotents, for example for those of the form $E_i = v_i^* v_i$, for unit row vectors v_i . In constructing constellations, using unitary matrices of the form $2E_i - I$, it is desirable to make the modulus of the determinants of the difference between any two E_i and E_j as large as possible.

4.2 Sets of real unitary constellations

Consider the following idempotents formed using vv^* for a row vector v . Assume now the entries of v are real. Each matrix is of the form $vv^* = \begin{pmatrix} |a|^2 & ab \\ ba & |b|^2 \end{pmatrix}$ where $|a|^2 + |b|^2 = 1$.

Let $v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ to give $E_1 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. This gives the unitary real matrix $A_1 = 2E_1 - I$.

Now let $a_2^2 = 1/3, b_2^2 = 2/3$ to give $E_2 = \begin{pmatrix} 1/3 & \sqrt{2}/3 \\ \sqrt{2}/3 & 2/3 \end{pmatrix}$. This gives the unitary real unitary matrix $A_2 = 2E_2 - I$.

In general let $a_k^2 = 1/(k+1), b_k^2 = k/(k+1)$ for $k \in \mathbb{N}$ to get $v_k = \begin{pmatrix} a_k \\ b_k \end{pmatrix}$ and then the real idempotent $E_k = \begin{pmatrix} \frac{1}{k+1} & \frac{\sqrt{k}}{k+1} \\ \frac{\sqrt{k}}{k+1} & \frac{k}{k+1} \end{pmatrix}$. Now let $A_k = 2E_k - I$ and then A_k is a unitary matrix.

Then consider $\mathcal{W} = \{A_k | k \geq 1\}$. This is an infinite set of unitary matrices. Constellations may be formed from subsets of \mathcal{W} . The difference between any two of the unitary matrices in \mathcal{V} is calculated as follows.

Proposition 4.2 Let $A_k, A_l \in \mathcal{V}$ where we assume $l > k$. Then $|\det(A_k - A_l)| = 4 \left(\frac{\sqrt{l} - \sqrt{k}}{\sqrt{(l+1)(k+1)}} \right)^2$.

Proof: $|\det(A_k - A_l)| = |\det 2(E_k - E_l)| = 4 \left(\frac{\sqrt{l} - \sqrt{k}}{\sqrt{(l+1)(k+1)}} \right)^2$ by Corollary 4.2. \square

Corollary 4.6 $\frac{1}{2} |\det(A_k - A_l)|^{\frac{1}{2}} = \frac{\sqrt{k} - \sqrt{l}}{\sqrt{(l+1)(k+1)}}$.

Corollary 4.7 $\mathcal{W} = \{A_k | k \geq 1\}$ is an infinite constellation with full diversity.

Further now extend the \mathcal{W} by taking the negatives of the elements which are also unitary matrices, that is let $\mathcal{X} = \{A_k, -A_k, |k \geq 1\} = \mathcal{W} \cup -\mathcal{W}$.

Proposition 4.3 $|\det(A_k + A_l)| = 4 \left(\frac{\sqrt{l} + \sqrt{k}}{\sqrt{(l+1)(k+1)}} \right)^2$.

Proof: Note that $A_k + A_l = 2E_k - I + 2E_l - I = 2E_k - 2(I - E_l)$. Now if E_l is of the form $\begin{pmatrix} a^2 & ab \\ ba & b^2 \end{pmatrix}$ with $a^2 + b^2 = 1$ then $I - E_l = \begin{pmatrix} b^2 & -ab \\ -ba & a^2 \end{pmatrix}$. Thus $I - E_l$ is the idempotent formed from $-b, a$ and apply Corollary 4.2 to get the result. \square

Proposition 4.7 below may also be used to show that $|\det(A_k + A_l)| \geq |\det(A_k - A_l)|$ which shows the quality is calculated from the differences $|\det(A_k - A_l)|$.

Corollary 4.8 $\frac{1}{2} |\det(A_k + A_l)|^{\frac{1}{2}} = \frac{\sqrt{l} + \sqrt{k}}{\sqrt{(l+1)(k+1)}}$.

Corollary 4.9 $\mathcal{X} = \{A_k, -A_k, |k \geq 1\} = \mathcal{W} \cup -\mathcal{W}$ is an infinite constellation with full diversity. The quality is the same as that of \mathcal{W} .

Note that the distance of A_k, A_l is smaller than the distance of $A_k, -A_l$. The quality may then be determined by the least of these in the constellation chosen from \mathcal{W} or from \mathcal{X} .

Finite constellations may be chosen from \mathcal{X} or \mathcal{W} and the quality may be directly calculated. This allows for a given rate R the construction of 2×2 constellations with full diversity and with this rate. Care should be taken in the choices from \mathcal{X} so as to ensure the quality is as large as possible. In Section 5 this will be extended to 4×4 , 8×8 constellations and so on.

Examples

- Consider $\mathcal{V} = \{A_1, A_2, A_3, A_4\} \cup \{-A_1, -A_2, -A_3, -A_4\}$. The rate here is $\log_2 8/2 = 1.5$. All the measured distances may be calculated and the least of these comes from $1/2|\det(A_3 - A_4)|^{1/2}$ which is approximately 0.05991526. To get better quality we need to take the unitary matrices to be ‘far enough apart’.
- Consider $\mathcal{V} = \{A_1, A_2, A_4, A_{16}\} \cup \{-A_1, -A_2, -A_4, -A_{16}\}$. The rate is again 1.5 and the smallest measured distance comes from A_4, A_2 giving the quality $\frac{\sqrt{16-\sqrt{4}}}{\sqrt{17*5}}$ which is approximately 0.217.
- Suppose it is required to construct a rate $R = 2$ constellation of 2×2 matrices. Suppose also it is required that the constellation consist of real matrices. Then choose $L = 2^4 = 16$ unitary matrices from \mathcal{X} or \mathcal{W} .

Consider $\{A_1, A_2, \dots, A_8, -A_1, -A_2, \dots, -A_8\}$. This has quality $\frac{\sqrt{8-\sqrt{7}}}{\sqrt{9*\sqrt{8}}}$. This is approximately 0.02153 but note the quality is given in terms of elements of quadratic extensions of \mathbb{Q} .

We can do better by spreading out the choice.

By using roots of unity as in Section 4.5 the quality may be increased to approximately 0.3826 while still maintaining non-commutativity.

More generally: For $p/q \in \mathbb{Q}$ with $p, q \in \mathbb{N}$, $p < q$ define $a_{p,q}^2 = p/q$, $b_{p,q}^2 = (q-p)/q$ and $v_{p,q} = \begin{pmatrix} a_{p,q} \\ b_{p,q} \end{pmatrix}$

to form the idempotent $E_{p,q} = \begin{pmatrix} \frac{p}{q} & \frac{\sqrt{p(q-p)}}{q} \\ \frac{\sqrt{p(q-p)}}{q} & \frac{q-p}{q} \end{pmatrix}$. Then define $A_{p,q} = 2E_{p,q} - I$ which is then a unitary matrix.

$$\text{For example } E_{4,7} = \begin{pmatrix} \frac{4}{7} & \frac{\sqrt{12}}{7} \\ \frac{\sqrt{12}}{7} & \frac{3}{7} \end{pmatrix}.$$

Constellations may then be formed from $\{A_{p,q}, -A_{p,q}\}$ by varying $p, q \in \mathbb{N}$, $q > p$. For example let $p = 1$, always, and vary q so as to give the previous system of constellations. Another example is let $p = 2$ and vary q with $q > 2$. The distances are relatively easy to work out. Which of these types are best needs to be worked on.

4.3 Real cos, sin

Take now $a = \cos \theta$, $b = \sin \theta$ to form $E = \begin{pmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{pmatrix} = \begin{pmatrix} \cos^2 \theta & \sin 2\theta/2 \\ \sin 2\theta/2 & \sin^2 \theta \end{pmatrix}$. Then let $A = 2E - I$.

Similarly form $F = \begin{pmatrix} \cos^2 \alpha & \cos \alpha \sin \alpha \\ \cos \alpha \sin \alpha & \sin^2 \alpha \end{pmatrix} = \begin{pmatrix} \cos^2 \alpha & \sin 2\alpha/2 \\ \sin 2\alpha/2 & \sin^2 \alpha \end{pmatrix}$. Then let $B = 2F - I$.

Then $|\det(A - B)| = |\det 2(E - F)| = 4(\cos \theta \sin \alpha - \sin \theta \cos \alpha)^2 = 4 \sin^2(\alpha - \theta)$.

Build the constellation from such unitary matrices. Thus it is required to build the constellation so that $|\sin(\theta - \alpha)|$ is as large as possible for all θ, α used in forming the constellation.

Define $E_{j,0} = \begin{pmatrix} \cos^2 \theta_j & \sin(\theta_j) \cos(\theta_j) \\ \cos(\theta_j) \sin(\theta_j) & \sin^2(\theta_j) \end{pmatrix}$ and $E_{j,1} = I - E_{j,0}$. Then let $U_j = E_{j,0} - E_{j,1} = 2E_{j,0} - I$ which is a unitary matrix.

Let the constellation then be defined as follows: Define $\theta_j = 2j\pi/n$ for $j = 0, 1, 2, \dots, n-1$ and $U_j = 2E_{j,0} - I$ be as above and the constellation is defined by $\mathcal{V}_n = \{U_0, U_1, \dots, U_{n-1}\}$.

Proposition 4.4 *For odd n the quality of the constellation \mathcal{V}_n is $|\sin(\pi/n)|$.*

Proof: By Corollary 4.2, for $A, B \in \mathcal{V}_n$, $A \neq B$, $|\det(A-B)| = 4|\sin(\theta_j - \theta_k)|^2$ for $j \neq k, 0 \leq j, k \leq n-1$. Thus $|\det(A-B)| = 4|\sin(2r\pi/n)|^2$ where $r = j-k$. This is never 0 for odd n and then it has minimum value $4|\sin((n+1)\pi/n)|^2 = 4|\sin(\pi/n)|^2$ attained when $r = (n+1)/2$, that is when $j = (n+1)/2, k = 0$ or $j = (n+3)/2, k = 1$ and others. Note that $|\sin(\pi + \alpha)| = |\sin \alpha|$ and that for $\alpha < \beta < \pi/2$ that $|\sin \beta| > |\sin \alpha|$.

The quality of the constellation (for odd n) is thus $\zeta_n = \frac{1}{2}(4|\sin^2(\pi/n)|)^{\frac{1}{2}} = |\sin(\pi/n)|$. \square

Thus for example:

$n = 5$: Rate $R = \log_2 5/2 = 1.1609..$, $\zeta_5 = (\sin(\pi/5)) = 0.58778....$ Of the 10 possible differences, 5 have difference 0.58778.. and 5 have difference 0.95105..

$n = 9$. Rate $R = \log_2 9/2 = 1.5849..$, $\zeta_9 = (\sin(\pi/9)) = 0.3420....$

$n = 17$. Rate $R = \log_2 17/2 = 2.0437....$, and quality $\zeta_{17} = \sin(\pi/17) = 0.1837..$

The constellation \mathcal{V}_n consists of real unitary matrices.

4.4 Extend the range

Consider the U_i constructed in Section 4.3. Note that $-U_j = 2(I - E_{0,j})$ is also a unitary matrix and we now include these with the constellation already constructed. Thus consider $\mathcal{X} = \mathcal{V}_n \cup -\mathcal{V}_n = \{U_0, U_1, \dots, U_{n-1}\} \cup \{-U_0, -U_1, \dots, -U_{n-1}\}$.

$U_j - (-U_j) = 2(U_j)$. Now $|\det(2(U_j))| = 4$. Then the difference of $U_j, -U_j$ is $\frac{1}{2}|\det(2(U_j))|^{1/2} = 1$.

As with Proposition 4.3 it is shown that $|\det(U_i + U_j)| > |\det(U_i - U_j)|$ so that the quality of the constellation is determined by the differences $|\det(U_i - U_j)|$.

Proposition 4.5 *For n odd the quality of \mathcal{X} is $|\sin(\pi/n)|$.*

Proof: This follows from Proposition 4.4 since $|\det(U_i + U_j)| > |\det(U_i - U_j)|$. \square

The rate is better here.

So for example when $n = 5$ get quality 0.58778.. and rate $\log_2(10)/2 = 1.6609...$

When $n = 9$ get quality $\sin(\pi/9) = 0.3420..$ and rate $\log_2(18)/2 = 2.0849..$

When $n = 17$ get quality $\sin(\pi/17) = 0.183749..$ and rate $\log_2(34)/2 = 2.5437...$

To get constellations with real entries of higher degree it is necessary to use constellations constructed from different sets of complete idempotents. Alternatively the 2×2 case can be upper loaded to 4×4 and then to 8×8 as shown in Section 5.

4.5 Range further

Suppose now a constellation $\mathcal{V} = \{A_1, A_2, \dots, A_t\}$ has been built. Assume the elements are real unitary matrices although this is not necessary in general.

We can extend the range as follows. For each j let $\mathcal{V}_j = \{A_j, \omega A_j, \omega^2 A_j, \dots, \omega^{k-1} A_j\}$ where ω is a primitive k^{th} root of unity and define $\mathcal{V}_\omega = \cup_{j=1}^k \mathcal{V}_j$. (The case $k = 2, \omega = -1$ was considered in Section 4.4.)

Then the following may be shown for unitary matrices constructed from the idempotents of type $\begin{pmatrix} aa^* & ab^* \\ ba^* & bb^* \end{pmatrix}$.

Proposition 4.6 *The quality of \mathcal{V}_ω is $\min\{\sin(\pi/k), \zeta\}$ where ζ is the quality of \mathcal{V} .*

Proof: Since $|\det(\omega^k A - \omega^j B)| = |\det(A - \omega^{j-k} B)|$ as $|\omega| = 1$ it is only necessary to look at differences $|\det(A - \omega^k B)|$. Now $|\det(A - \omega^j A)| = |\det(1 - \omega^j A)| = |(1 - \omega^j)^2 \det A| = |(1 - \omega^j)^2|$ since A is unitary. Now $|(1 - \omega^j)^2|$ is least when $j = 1$. We are interested then in the difference $1/2(|1 - \omega|^2)^{\frac{1}{2}} = 1/2|1 - \omega|$. Now $1/2|1 - \omega| = 1/2|1 - e^{2\pi i/k}| = 1/2|\sqrt{2 - 2\cos 2\pi/k}| = 1/2|\sqrt{4\sin^2 \pi/k}| = |\sin \pi/k|$.

Now consider $|\det(A - \omega^j B)|$. By Proposition 4.1 this is $|\omega \det(A - B)| = |\det(A - B)|$. Thus $\mathcal{V}_\omega = \min\{\sin(\pi/k), \zeta\}$. □

Now apply this result to get examples of constellations as follows.

1. Consider the constellation $\{B_1 = A_1, B_2 = A_2, B_3 = A_4, B_4 = A_{16}\}$ as in Section 4.2. As was shown in that Section, this has quality $\zeta = \frac{\sqrt{16}-\sqrt{4}}{\sqrt{85}}$ which is approximately 0.217. Now extend this to the constellation $\mathcal{V} = \{B_1, B_2, B_3, B_4, \omega B_1, \omega B_2, \omega B_3, \omega B_4, \omega^2 B_1, \omega^2 B_2, \omega^2 B_3, \omega^2 B_4, \omega^3 B_1, \omega^3 B_2, \omega^3 B_3, \omega^3 B_4\}$ where ω is a primitive 4^{th} of unity. This is a constellation of sixteen 2×2 matrices which by Proposition 4.6 has quality $\min\{\zeta, \sin(\pi/4)\} = \zeta \approx 0.217$. The rate is $\log_2 16/2 = 2$.
2. This is similar to previous example 1. except now let ω be an 8^{th} root of unity and consider the constellation of 32 unitary matrices obtained in this way. The rate is $\log_2(32)/2 = 2.5$ and the quality is $\min\{\zeta, \sin \pi/8\} = \zeta$ as $\sin(\pi/8) \approx 0.38268...$
3. Let now ω be a primitive 16^{th} root of unity and then as in example 1. get a constellation of 64 unitary matrices with rate $\log_2(64)/2 = 3$ and quality $\min\{\zeta, \sin(\pi/16)\} = \sin(\pi/16) \approx 0.1950$.

The above are just a few of the constellations that may be constructed by this method.

Once 2×2 constellations are constructed they may be used to get 4×4 , then 8×8 constellations etc. by the methods of Section 5 and these new ones will have similar quality and rate.

4.6 Constellations constructed from complex symmetric idempotents

In Sections 4.2 and 4.3, real unitary matrices are constructed from orthogonal sets of idempotents. Now complex constellations are constructed from complex idempotents.

We specialise to 2×2 unitary matrices formed from complex symmetric orthogonal idempotents although other cases may also be considered.

We construct 2×2 unitary matrices from the idempotents uu^* where u is a unit vector and may have complex entries. Then as noted $E = uu^* = \begin{pmatrix} aa^* & ab^* \\ ba^* & bb^* \end{pmatrix}$ and $aa^* + bb^* = 1$.

The unitary matrix $2E - I$ is formed. The differences between such unitary may be calculated from Corollary 4.2.

Let $v = \begin{pmatrix} a \\ b \end{pmatrix}$ be a unit vector in \mathbb{C}_2 . Form $E_{a,b} = vv^*$ which is then an idempotent and let $A_{a,b} = 2E_{a,b} - I$ which is a unitary matrix.

Constellations are then formed from such unitary matrices and the differences are calculated from Corollary 4.2. This is a very general construction and there is no restriction on $v \in \mathbb{C}_2$ except that it be a unit vector.

Now consider constellations which will have entries in $\mathbb{Q}(i)$. Consider a vector $u = \begin{pmatrix} a \\ b \end{pmatrix}$ with $a, b \in \mathbb{Z}(i)$ where $|u| = \sqrt{t}$ with $t \in \mathbb{N}$. Then $v = \frac{1}{\sqrt{t}}u$ is a unit vector and the idempotent formed from v has the form $E = \frac{1}{t} \begin{pmatrix} aa^* & ab^* \\ ba^* & bb^* \end{pmatrix}$ and the corresponding unitary matrix has the form $2E - I$ which has entries in $\mathbb{Q}(i)$. We refer to this matrix as $A_{a,b}$ although $(c, d) = \alpha(a, b)$ will produce the same unitary matrix.

Form constellations of the form $\{A_{a_j, b_j} | j \in J, a_j, b_j \in \mathbb{Z}, (a_k, b_k) \neq \alpha(a_j, b_j)k \neq j\}$.

For example:

$$(1) a = 1 + 2i, b = 2 + i, E_1 = \frac{1}{10} \begin{pmatrix} 5 & 4 + 3i \\ 4 - 3i & 5 \end{pmatrix}, (2) a = 1 + 3i, b = 3 + i, E_2 = \frac{1}{20} \begin{pmatrix} 10 & 6 + 8i \\ 6 - 8i & 10 \end{pmatrix},$$

(3) $a = 2 + 3i, b = 3 + i, E_3 = \frac{1}{26} \begin{pmatrix} 13 & 12 + 5i \\ 12 - 5i & 5 \end{pmatrix}$, (4): $a = 2 + 3i, b = 1 + i, E_4 = \frac{1}{15} \begin{pmatrix} 13 & 5 + i \\ 5 - i & 2 \end{pmatrix}$.

Let $A_i = 2E_i - I$ and form the constellation $\{A_1, A_2, A_3, A_4, -A_1, -A_2, -A_3, -A_4\}$. The quality of the constellation is easily worked out using Corollary 4.2 and is left as an exercise.

4.7 Extending the range

Here it is shown how to extend constellations which may be done without loss of quality.

Lemma 4.3 *Let A be a unitary $m \times m$ matrix and $\omega = e^{i\theta}$ a complex number of modulus 1. Then $\frac{1}{2} |\det(A - \omega A)|^{\frac{1}{m}} = |\sin(\frac{\theta}{2})|$.*

Proof: $\frac{1}{2} |\det(A - \omega A)|^{\frac{1}{m}} = \frac{1}{2} |\det((1 - \omega)A)|^{\frac{1}{m}} = \frac{1}{2} |(1 - \omega)^m \det(A)|^{\frac{1}{m}} = \frac{1}{2} |(1 - \omega)^m|^{\frac{1}{m}} = \frac{1}{2} |1 - \omega| = \frac{1}{2} |\sqrt{(1 - \omega)(1 - \omega^*)}| = \frac{1}{2} |\sqrt{2 - (\omega + \omega^*)}| = \frac{1}{2} |\sqrt{2(1 - \cos(\theta))}| = \frac{1}{2} |\sqrt{2 * 2 \sin^2(\frac{\theta}{2})}| = |\sin(\frac{\theta}{2})|$. \square

Proposition 4.7 *Let A, B be unitary matrices and $\omega = e^{i\theta}$ a complex number of modulus 1. Then $|\det(A - \omega B)| \geq |\det(A - B)|$.*

Proof: Now $|\det(A - \omega B)| = |\det(A(I - \omega A^* B))| = |\det(A) \det(I - \omega A^* B)| = |\det(A)| |\det(I - \omega A^* B)| = |\det(I - \omega A^* B)|$ as A is unitary. Similarly $|\det(A - B)| = |\det(I - A^* B)|$.

Thus it is only necessary to show $|\det(I - \omega X)| \geq |\det(I - X)|$ for a unitary matrix X .

Let k be the size of the matrices. As X is unitary there exists a unitary matrix P such that $P^* X P = D$ where D is a diagonal matrix with diagonal entries $\{d_1, d_2, \dots, d_k\}$. Then also $P^*(I - X)P = D_0$ where D_0 is diagonal with diagonal entries $\{1 - d_1, 1 - d_2, \dots, 1 - d_k\}$ and $P^*(I - \omega X) = D_1$ where D_1 is diagonal with diagonal entries $\{1 - \omega d_1, 1 - \omega d_2, \dots, 1 - \omega d_k\}$.

Then $|\det(I - X)| = |\prod_{i=1}^k (1 - d_i)| = \prod_{i=1}^k |1 - d_i|$ and $|\det(I - \omega X)| = |\prod_{i=1}^k (1 - \omega d_i)| = \prod_{i=1}^k |1 - \omega d_i|$.

For complex numbers $|z_1 - z_2| \geq ||z_1| - |z_2||$. Let $z_1 = 1, z_2 = \omega d$ and then $|1 - \omega d| \geq ||1| - |\omega d|| = |1 - d|$ as $|\omega| = 1$. Thus $|1 - \omega d_i| \geq |1 - d_i|$ for each i and so $|\det(I - \omega X)| \geq |\det(I - X)|$. \square

Proposition 4.8 *Let $\mathcal{V} = \{A_1, A_2, \dots, A_n\}$ be a fully diverse constellation of n matrices with quality ζ and $\omega = e^{2\pi i/k}$ a primitive k^{th} root of unity. Define $\mathcal{V}_{i,\omega} = \{A_i, \omega A_i, \dots, \omega^{k-1} A_i\}$ for $i = 1, 2, \dots, n$ and $\mathcal{V}_\omega = \cup_{i=1}^n \mathcal{V}_{i,\omega}$. Suppose $A_j \neq \omega^t A_l$ for $j \neq l$ and for $1 \leq t \leq k - 1$. Then the quality of \mathcal{V}_ω is $\min\{\zeta, |\sin(\frac{\pi}{k})|\}$.*

Proof: The result follows from Proposition 4.7 and Lemma 4.3. Note that $|\det(\omega^k A - \omega^j B)| = |\det(A - \omega^{j-k} B)|$ and that $|\sin(\frac{r\pi}{k})| \geq |\sin(\frac{\pi}{k})|$ for $1 \leq r \leq k - 1$. \square

This enables the construction of a constellation with kn elements from a constellation \mathcal{V} with n elements and the quality is the same provided the quality of \mathcal{V} is greater than or equal to $\sin(\pi/k)$.

5 Tangle to construct higher order constellations

In [9] and [10] the idea of a *tangle of matrices* is introduced. This construction is now used to construct constellations of matrices of higher order from constellations of smaller order matrices.

Suppose A, B are matrices of the same size. Then a *tangle* of $\{A, B\}$ is one of

1. $W = \frac{1}{\sqrt{2}} \begin{pmatrix} A & A \\ B & -B \end{pmatrix}$.
2. $W = \frac{1}{\sqrt{2}} \begin{pmatrix} A & B \\ A & -B \end{pmatrix}$.

Note that 2. is the transpose of 1.

A tangle of $\{A, B\}$ is not the same as, and is not necessarily equivalent to, a tangle of $\{B, A\}$ which is one of $\frac{1}{\sqrt{2}} \begin{pmatrix} B & A \\ B & -A \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} B & -A \\ B & A \end{pmatrix}$.

Note that interchanging any rows and/or columns of a unitary matrix results in a unitary matrix.

If $A = B$ then a tangle of $\{A, A\}$ is a tensor product but a tangle of $\{A, B\}$ is not necessarily a tensor product when $A \neq B$; this is why they can be useful for constructions.

Then as in [10] or [9] the following may be shown.¹

Proposition 5.1 *Let A, B be unitary matrices of the same size. Then a tangle of $\{A, B\}$ or of $\{B, A\}$ is a unitary matrix.*

Proof: This is shown for $W = \frac{1}{\sqrt{2}} \begin{pmatrix} A & A \\ B & -B \end{pmatrix}$; the proofs for the others are similar. Now $WW^* = \frac{1}{\sqrt{2}} \begin{pmatrix} A & A \\ B & -B \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} A^* & B^* \\ A^* & -B^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} AA^* + AA^* & AB^* - AB^* \\ BA^* - BA^* & BB^* + BB^* \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2I_n & \underline{0} \\ \underline{0} & 2I_n \end{pmatrix} = I_{2n}$. \square

Lemma 5.1 *Let A be an $n \times n$ matrix. Then $\det(\alpha A) = \alpha^n \det(A)$ for a scalar α .*

Given a constellation of size $n \times n$ there are a number of ways of constructing constellations of size $2n \times 2n$ from a constellation of size $n \times n$ using tangled products.

Here is an example to explain the method in general. Let $\mathcal{V}_0 = \{A_1, A_2, A_3, A_4, -A_1, -A_2, -A_3, -A_4\}$ be a constellation of $m \times m$ unitary matrices as for example constructed in Sections 4.4 or 4.6.

Then consider the following constellation of $2m \times 2m$ matrices.

$$\begin{aligned} \mathcal{V} = & \frac{1}{\sqrt{2}} \begin{pmatrix} A_1 & A_1 \\ A_1 & -A_1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} A_2 & A_2 \\ A_2 & -A_2 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} A_3 & A_3 \\ A_3 & -A_3 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} A_4 & A_4 \\ A_4 & -A_4 \end{pmatrix}, \\ & \frac{1}{\sqrt{2}} \begin{pmatrix} -A_1 & -A_1 \\ -A_1 & A_1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -A_2 & -A_2 \\ -A_2 & A_2 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -A_3 & -A_3 \\ -A_3 & A_3 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -A_4 & -A_4 \\ -A_4 & A_4 \end{pmatrix}, \\ & \frac{1}{\sqrt{2}} \begin{pmatrix} A_1 & -A_1 \\ A_2 & A_2 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} A_2 & -A_2 \\ A_1 & A_1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} A_3 & -A_3 \\ A_4 & A_4 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} A_4 & -A_4 \\ A_3 & A_3 \end{pmatrix}, \\ & \frac{1}{\sqrt{2}} \begin{pmatrix} -A_1 & A_1 \\ -A_2 & -A_2 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -A_2 & A_2 \\ -A_1 & -A_1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -A_3 & A_3 \\ -A_4 & -A_4 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -A_4 & A_4 \\ -A_3 & -A_3 \end{pmatrix} \end{aligned}$$

The first 8 could be considered as tensor products. Notice that the second 4 are the negatives of the first four but they could also be considered as tangles of the negatives $-A_1, -A_2, -A_3, -A_4$.

Proposition 5.2 *Suppose \mathcal{V}_0 has quality ζ . Then the quality of \mathcal{V} is $2^{\frac{1}{4}}\zeta$.*

Proof: The proof consists of working out the differences. We show the proof when the blocks are of size 2×2 ; the other cases are similar.

Consider the difference of the matrices

$$\begin{aligned} & \left| \det\left(\frac{1}{\sqrt{2}} \begin{pmatrix} A_1 & A_1 \\ A_1 & -A_1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} A_2 & A_2 \\ A_2 & -A_2 \end{pmatrix} \right) \right| \\ &= \left| \det\left(\frac{1}{\sqrt{2}} \begin{pmatrix} A_1 - A_2 & A_1 - A_2 \\ A_1 - A_2 & -A_1 + A_2 \end{pmatrix} \right) \right| \\ &= \left| \det\left(\frac{1}{\sqrt{2}}(A_1 - A_2)(-A_1 + A_2) - (A_1 - A_2)(A_1 - A_2)\right) \right| \end{aligned}$$

since $(A_1 - A_2)$ and $(-A_1 + A_2)$ commute.

Thus this difference $\delta = \left| \det\left(\frac{1}{\sqrt{2}}2(A_1 - A_2)^2\right) \right| = \left| \det\left(\frac{2}{\sqrt{2}}(A_1 - A_2)^2\right) \right| = \left| \frac{4}{2}(\det(A_1 - A_2)^2) \right| = 2|\det(A_1 - A_2)^2|$, by Lemma 5.1 as the A_i are 2×2 matrices.

Now it is known that $\frac{1}{2}|\det(A_1 - A_2)|^{1/2} \geq \zeta$ and so $\frac{1}{2}\delta^{\frac{1}{4}} = \frac{1}{2}(2|\det(A_1 - A_2)^2|)^{1/4} = \frac{1}{2}2^{\frac{1}{4}}|\det(A_1 - A_2)|^{\frac{1}{2}} \geq 2^{\frac{1}{4}}\zeta$.

Similarly the other differences are shown to be $\geq 2^{\frac{1}{4}}\zeta$.

Note that $d^{\frac{1}{2}} \geq d$ when $0 \leq d \leq 1$; this is needed for some of the other difference calculations.

¹The result in [10] and [9] is more general where it is shown for *paraunitary matrices*; unitary matrices are special cases of paraunitary matrices.

It is clear also, since \mathcal{V}_0 contains the exact difference ζ , that the difference $2^{1/4}\zeta$ is attained by \mathcal{V} . \square

Thus in this manner it is possible to start out with a constellation of eight 2×2 matrices of quality ζ , then construct a constellation of sixteen 4×4 matrices of quality $2^{1/4}\zeta$ from these construct a constellation of thirty-two 8×8 matrices of quality $2^{1/4}2^{1/4}\zeta$ and so on. The rates go from $\log_2 8/2 = 1.5$ to $\log_2 16/4 = 1$ to $\log_2 32/8 = 0.625$ with higher order and the quality goes up slightly. By starting out with 16 in the original constellation the rates go from 2 to 1.25 to .75 with higher order.

5.1 A general construction using tangles

More generally proceed as follows. Consider a constellation $\mathcal{V}_0 = \{A_1, A_2, A_3, \dots, A_k\}$ of $n \times n$ matrices such that $A_i \neq \omega^s A_j$ for $i \neq j$ where ω is a primitive t^{th} root of unity. Consider $k = 2w$ to make the explanation slightly simpler but this isn't necessary. Let $B_i = \frac{1}{2} \begin{pmatrix} A_i & A_i \\ A_i & -A_i \end{pmatrix}$ for $i = 1, 2, \dots, 2w$ and then for $i = 1, 3, \dots, 2w - 1$ define $C_i = \frac{1}{2} \begin{pmatrix} A_i & -A_i \\ A_{i+1} & A_{i+1} \end{pmatrix}, D_i = \frac{1}{2} \begin{pmatrix} A_{i+1} & -A_{i+1} \\ A_i & A_i \end{pmatrix}$. Let $\mathcal{V}_i = \{B_i, \omega B_i, \omega^2 B_i, \dots, \omega^{t-1} B_i\}$ for $i = 1, \dots, 2w$ and $\mathcal{W}_i = \{C_i, \omega C_i, \dots, \omega^{t-1} C_i, D_i, \omega D_i, \dots, \omega^{t-1} D_i\}$ for $i = 1, 3, \dots, 2w - 1$.

Let $\mathcal{V}_{0,\omega} = \bigcup_{i=1}^{2w} \mathcal{V}_i \cup_{k=0}^{w-1} \mathcal{W}_{2k+1}$. Now $\mathcal{V}_{0,\omega}$ consists of $2n \times 2n$ unitary matrices and has $4kt$ elements.

Proposition 5.3 *The quality of $\mathcal{V}_{0,\omega}$ is $\min\{2^{1/4}\zeta, \sin(\pi/t)\}$ where ζ is the quality of \mathcal{V}_0 .*

The proof is omitted but depends on Propositions 4.7 and 4.8 and calculations similar to those in Proposition 5.2.

Samples Let $\mathcal{V}_0 = \{A_1, A_2, A_4, A_{16}\}$ be a constellation of 2×2 matrices with quality $\zeta \approx 0.217$, as in Section 4.2, and ω a primitive 8^{th} root of unity. Then $\mathcal{V}_{0,\omega}$ has quality $\min\{2^{1/4}\zeta, \sin(\pi/8)\}$ and rate $\log_2 64/4 = 3$. Now $\sin(\pi/8) \approx 0.3827$ so $2^{1/4}\zeta \approx 0.258$ is the quality.

Consider the elements in $\mathcal{V}_{0,\omega}$ not involving ω and these form a constellation \mathcal{W}_0 of 8 unitary 4×4 matrices. Now form $\mathcal{W}_{0,\omega}$ where ω is again a primitive 8^{th} root of unity. This gives a constellation of 128 unitary 8×8 matrices which has quality $\min\{2^{1/4}2^{1/4}\zeta, \sin(\pi/8)\}$. Now $2^{1/2}\zeta \approx 0.3069$ and this is the quality. The rate is $\log_2(128)/8 = 0.875$.

General conclusion

The methods allow the construction of constellations of many types and sizes and the quality may be calculated directly. In many cases the complete set of differences can be worked out as required. Idempotents are building blocks for unitary matrices. The samples given within are a small subset of the possibilities and many more fully diverse constellations may be developed by the methods.

References

- [1] A. Shokrollahi, B. Hassibi, B.M. Hochwald, W. Sweldens, "Representation theory for high-rate multiple-antenna code design", IEEE Trans. on Inform. Theory, 47, no.6, (2001), 2335-2367.
- [2] Grégory Berhuy and Frédérique Oggier, *An Introduction to Central Simple Algebras and Their Applications to Wireless Communications*, AMS Mathematical Surveys and Monographs, Vol 191, Providence RI, 2013.
- [3] *Channel Coding: Theory, Algorithms, and Applications*, Edited by: David Declerq, Marc Fossorier and Ezio Biglieri, Academic Press Library in Mobile and Wireless Communications, Chapter 10, 2014.
- [4] B. Hochwald, W. Sweldens, "Differential unitary space time modulation", IEEE Trans. Comm., 48, (2000), 2041-2052.

- [5] B.A. Sethuraman, “Division Algebras and Wireless Communication”, Notices of the AMS, 57, no. 11 (2010), 1432-1439.
- [6] Richard E. Blahut, *Algebraic Codes for data transmission*, Cambridge University Press, 2003.
- [7] I. Kovacs, D.S. Silver, S.G. Williams, “Determinants of Commuting-Block matrices”, Amer. Math. Monthly, 106, no. 10, 950-952, 1999.
- [8] Paul Hurley and Ted Hurley, “Codes from zero-divisors and units in group rings”, Int. J. Inform. and Coding Theory, 1 (2009), 57-87.
- [9] Barry Hurley and Ted Hurley, “Paraunitary matrices and group rings”, Intn. J. Group Theory, 3, no.1, 31-56, 2014.
- [10] Barry Hurley and Ted Hurley, “Paraunitary matrices”, arXiv:1205.0703v1.
- [11] Oskar M. Baksalary, Dennis S. Bernstein, Gtz Trenkler, “On the equality between rank and trace of an idempotent matrix”, Applied Mathematics and Computation, 217, 4076-4080, 2010.
- [12] César Milies & Sudarshan Sehgal, *An introduction to Group Rings*, Klumer, 2002.